9.2 Euler's Method

The reader should be convinced that not all initial value problems can be solved explicitly, and often it is impossible to find a formula for the solution $y(t)$; for example, there is no "closed-form expression" for the solution to $y' = t^3 + y^2$ with $y(0) = 0$. Hence for engineering and scientific purposes it is necessary to have methods for approximating the solution. If a solution with many significant digits is required, then more computing effort and a sophisticated algorithm must be used.

The first approach, called Euler's method, serves to illustrate the concepts involved in the advanced methods. It has limited use because of the larger error that is accumulated as the process proceeds. However, it is important to study because the error analysis is easier to understand.

Let $[a, b]$ be the interval over which we want to find the solution to the well-posed I.V.P. $y' = f(t, y)$ with $y(a) = y_0$. In actuality, we will not find a differentiable function that satisfies the I.V.P. Instead, a set of points $\{(t_k, y_k)\}$ is generated, and the points are used for an approximation (i.e., $y(t_k) \approx y_k$). How can we proceed to construct a "set of points" that will "satisfy a differential equation approximately"? First we choose the abscissas for the points. For convenience we subdivide the interval $[a, b]$ into M equal subintervals and select the mesh points

$$(1) \qquad t_k = a + kh \quad \text{for } k = 0, 1, \ldots, M \text{ where } h = \frac{b-a}{M}.$$

The value h is called the ***step size***. We now proceed to solve approximately

$$(2) \qquad y' = f(t, y) \quad \text{over} \quad [t_0, t_M] \quad \text{with} \quad y(t_0) = y_0.$$

Assume that $y(t)$, $y'(t)$, and $y''(t)$ are continuous and use Taylor's theorem to expand $y(t)$ about $t = t_0$. For each value t there exists a value c_1 that lies between t_0 and t so that

$$(3) \qquad y(t) = y(t_0) + y'(t_0)(t - t_0) + \frac{y''(c_1)(t - t_0)^2}{2}.$$

When $y'(t_0) = f(t_0, y(t_0))$ and $h = t_1 - t_0$ are substituted in equation (3), the result is an expression for $y(t_1)$:

$$y(t_1) = y(t_0) + hf(t_0, y(t_0)) + y''(c_1)\frac{h^2}{2}. \tag{4}$$

If the step size h is chosen small enough, then we may neglect the second-order term (involving h^2) and get

$$y_1 = y_0 + hf(t_0, y_0), \tag{5}$$

which is ***Euler's approximation***.

The process is repeated and generates a sequence of points that approximates the solution curve $y = y(t)$. The general step for Euler's method is

$$t_{k+1} = t_k + h, \qquad y_{k+1} = y_k + hf(t_k, y_k) \qquad \text{for } k = 0, 1, \ldots, M-1. \tag{6}$$

Example 9.2. Use Euler's method to solve approximately the initial value problem

$$y' = Ry \qquad \text{over } [0, 1] \text{ with } y(0) = y_0 \text{ and } R \text{ constant.} \tag{7}$$

The step size must be chosen, and then the second formula in (6) can be determined for computing the ordinates. This formula is sometimes called a *difference equation*, and in this case it is

$$y_{k+1} = y_k(1 + hR) \qquad \text{for } k = 0, 1, \ldots, M-1. \tag{8}$$

If we trace the solution values recursively, we see that

$$\begin{aligned} y_1 &= y_0(1 + hR) \\ y_2 &= y_1(1 + hR) = y_0(1 + hR)^2 \\ &\vdots \\ y_M &= y_{M-1}(1 + hR) = y_0(1 + hR)^M. \end{aligned} \tag{9}$$

Table 9.1 Compound Interest in Example 9.3

Step size, h	Number of iterations, M	Approximation to $y(5)$, y_M
1	5	$1000\left(1+\frac{0.1}{1}\right)^{5} = 1610.51$
$\frac{1}{12}$	60	$1000\left(1+\frac{0.1}{12}\right)^{60} = 1645.31$
$\frac{1}{360}$	1800	$1000\left(1+\frac{0.1}{360}\right)^{1800} = 1648.61$

For most problems there is no explicit formula for determining the solution points, and each new point must be computed successively from the previous point. However, for the initial value problem (7) we are fortunate; Euler's method has the explicit solution

$$t_k = kh \qquad y_k = y_0(1+hR)^k \qquad \text{for } k = 0,\ 1,\ \ldots,\ M. \tag{10}$$

Formula (10) can be viewed as the "compound interest" formula, and the Euler approximation gives the future value of a deposit. ■

Example 9.3. Suppose that $1000 is deposited and earns 10% interest compounded continuously over 5 years. What is the value at the end of 5 years?

We choose to use Euler approximations with $h = 1$, $\frac{1}{12}$, and $\frac{1}{360}$ to approximate $y(5)$ for the I.V.P.:

$$y' = 0.1y \qquad \text{over } [0, 5] \text{ with } y(0) = 1000.$$

Formula (10) with $R = 0.1$ produces Table 9.1. ■

Think about the different values y_5, y_{60}, and y_{1800} that are used to determine the future value after 5 years. These values are obtained using different step sizes and reflect different amounts of computing effort to obtain an approximation to $y(5)$. The solution to the I.V.P. is $y(5) = 1000e^{0.5} = 1648.72$. If we did not use the closed-form solution (10), then it would have required 1800 iterations of Euler's method to obtain y_{1800}, and we still have only five digits of accuracy in the answer!

If bankers had to approximate the solution to the I.V.P. (7), they would choose Euler's method because of the explicit formula in (10). The more sophisticated methods for approximating solutions do not have an explicit formula for finding y_k, but they will require less computing effort.

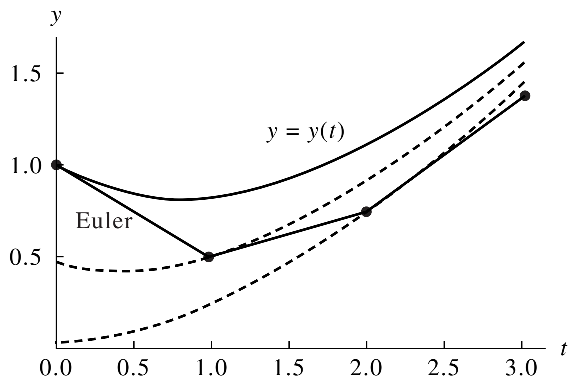

Figure 9.5 Euler's approximations $y_{k-1} = y_k + hf(t_k, y_k)$.

Geometric Description

If you start at the point (t_0, y_0) and compute the value of the slope $m_0 = f(t_0, y_0)$ and move horizontally the amount h and vertically $hf(t_0, y_0)$, then you are moving along the tangent line to $y(t)$ and will end up at the point (t_1, y_1) (see Figure 9.5). Notice that (t_1, y_1) is not on the desired solution curve! But this is the approximation that we are generating. Hence we must use (t_1, y_1) as though it were correct and proceed by computing the slope $m_1 = f(t_1, y_1)$ and using it to obtain the next vertical displacement $hf(t_1, y_1)$ to locate (t_2, y_2), and so on.

Step Size versus Error

The methods we introduce for approximating the solution of an initial value problem are called ***difference methods*** or ***discrete variable methods***. The solution is approximated at a set of discrete points called a *grid* (or *mesh*) of points. An elementary single-step method has the form $y_{k+1} = y_k + h\Phi(t_k, y_k)$ for some function Φ called an ***increment function***.

When using any discrete variable method to solve an initial value problem approximately, there are two sources of error: discretization and round off.

Definition 9.3. Assume that $\{(t_k, y_k)\}_{k=0}^{M}$ is the set of discrete approximations and that $y = y(t)$ is the unique solution to the initial value problem.

The ***global discretization error*** e_k is defined by

$$e_k = y(t_k) - y_k \qquad \text{for } k = 0, 1, \ldots, M. \tag{11}$$

It is the difference between the unique solution and the solution obtained by the discrete variable method.

The ***local discretization error*** ϵ_{k+1} is defined by

$$\epsilon_{k+1} = y(t_{k+1}) - y_k - h\Phi(t_k, y_k) \qquad \text{for } k = 0, 1, \ldots, M - 1. \tag{12}$$

It is the error committed in the single step from t_k to t_{k+1}. ▲

When we obtained equation (6) for Euler's method, the neglected term for each step was $y^{(2)}(c_k)(h^2/2)$. If this was the only error at each step, then at the end of the interval $[a, b]$, after M steps have been made, the accumulated error would be

$$\sum_{k=1}^{M} y^{(2)}(c_k)\frac{h^2}{2} \approx My^{(2)}(c)\frac{h^2}{2} = \frac{hM}{2}y^{(2)}(c)h = \frac{(b-a)y^{(2)}(c)}{2}h = \boldsymbol{O}(h^1).$$

There could be more error, but this estimate predominates. A detailed discussion on this topic can be found in advanced texts on numerical methods for differential equations.

Theorem 9.3 (Precision of Euler's Method). Assume that $y(t)$ is the solution to the I.V.P. given in (2). If $y(t) \in C^2[t_0, b]$ and $\{(t_k, y_k)\}_{k=0}^{M}$ is the sequence of approximations generated by Euler's method, then

$$|e_k| = |y(t_k) - y_k| = \boldsymbol{O}(h),$$
$$|\epsilon_{k+1}| = |y(t_{k+1}) - y_k - hf(t_k, y_k)| = \boldsymbol{O}(h^2). \tag{13}$$

The error at the end of the interval is called the ***final global error (F.G.E.)***:

$$E\,(y(b), h) = |y(b) - y_M| = \boldsymbol{O}(h). \tag{14}$$

Remark. The final global error $E(y(b), h)$ is used to study the behavior of the error for various step sizes. It can be used to give us an idea of how much computing effort must be done to obtain an accurate approximation.

Examples 9.4 and 9.5 illustrate the concepts in Theorem 9.3. If approximations are computed using the step sizes h and $h/2$, we should have

$$E(y(b), h) \approx Ch \tag{15}$$

for the larger step size, and

$$E\left(y(b), \frac{h}{2}\right) \approx C\frac{h}{2} = \frac{1}{2}Ch \approx \frac{1}{2}E(y(b), h). \tag{16}$$

Hence the idea in Theorem 9.3 is that if the step size in Euler's method is reduced by a factor of $\frac{1}{2}$, we can expect that the overall F.G.E. will be reduced by a factor of $\frac{1}{2}$.

Example 9.4. Use Euler's method to solve the I.V.P.

$$y' = \frac{t-y}{2} \quad \text{on } [0, 3] \text{ with } y(0) = 1.$$

Compare solutions for $h = 1, \frac{1}{2}, \frac{1}{4}$, and $\frac{1}{8}$.

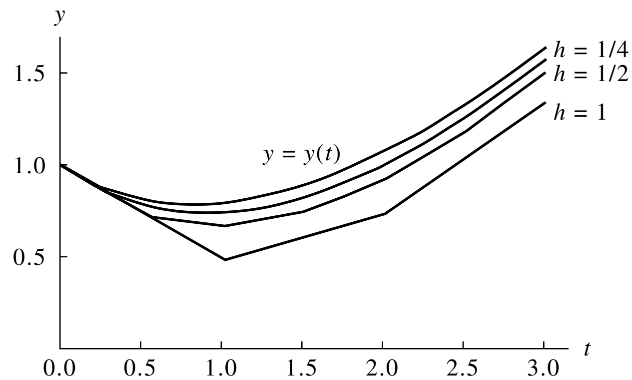

Figure 9.6 Comparison of Euler solutions with different step sizes for $y' = (t - y)/2$ over $[0, 3]$ with the initial condition $y(0) = 1$.

Figure 9.6 shows graphs of the four Euler solutions and the exact solution curve $y(t) = 3e^{-t/2} - 2 + t$. Table 9.2 gives the values for the four solutions at selected abscissas. For the step size $h = 0.25$, the calculations are

$$y_1 = 1.0 + 0.25\left(\frac{0.0 - 1.0}{2}\right) = 0.875,$$

$$y_2 = 0.875 + 0.25\left(\frac{0.25 - 0.875}{2}\right) = 0.796875, \qquad \text{etc.}$$

This iteration continues until we arrive at the last step:

$$y(3) \approx y_{12} = 1.440573 + 0.25\left(\frac{2.75 - 1.440573}{2}\right) = 1.604252. \qquad \blacksquare$$

Example 9.5. Compare the F.G.E. when Euler's method is used to solve the I.V.P.

$$y' = \frac{t - y}{2} \quad \text{over } [0, 3] \text{ with } y(0) = 1,$$

using step sizes $1, \frac{1}{2}, \ldots, \frac{1}{64}$.

Table 9.3 gives the F.G.E. for several step sizes and shows that the error in the approximation to $y(3)$ decreases by about $\frac{1}{2}$ when the step size is reduced by a factor of $\frac{1}{2}$. For the smaller step sizes the conclusion of Theorem 9.3 is easy to see:

$$E(y(3), h) = y(3) - y_M = \boldsymbol{O}(h^1) \approx Ch, \qquad \text{where} \qquad C = 0.256. \qquad \blacksquare$$

Table 9.2 Comparison of Euler Solutions with Different Step Sizes for $y' = (t - y)/2$ over [0, 3] with $y(0) = 1$

t_k	y_k				$y(t_k)$ Exact
	$h = 1$	$h = \frac{1}{2}$	$h = \frac{1}{4}$	$h = \frac{1}{8}$	
0	1.0	1.0	1.0	1.0	1.0
0.125				0.9375	0.943239
0.25			0.875	0.886719	0.897491
0.375				0.846924	0.862087
0.50		0.75	0.796875	0.817429	0.836402
0.75			0.759766	0.786802	0.811868
1.00	0.5	0.6875	0.758545	0.790158	0.819592
1.50		0.765625	0.846386	0.882855	0.917100
2.00	0.75	0.949219	1.030827	1.068222	1.103638
2.50		1.211914	1.289227	1.325176	1.359514
3.00	1.375	1.533936	1.604252	1.637429	1.669390

Table 9.3 Relation between Step Size and F.G.E. for Euler Solutions to $y' = (t - y)/2$ over [0, 3] with $y(0) = 1$

Step size, h	Number of steps, M	Approximation to $y(3)$, y_M	F.G.E. Error at $t = 3$, $y(3) - y_M$	$\boldsymbol{O}(h) \approx Ch$ where $C = 0.256$
1	3	1.375	0.294390	0.256
$\frac{1}{2}$	6	1.533936	0.135454	0.128
$\frac{1}{4}$	12	1.604252	0.065138	0.064
$\frac{1}{8}$	24	1.637429	0.031961	0.032
$\frac{1}{16}$	48	1.653557	0.015833	0.016
$\frac{1}{32}$	96	1.661510	0.007880	0.008
$\frac{1}{64}$	192	1.665459	0.003931	0.004

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