

Theorem 2.1. Assume that g is a continuous function and that $\{p_n\}_{n=0}^{\infty}$ is a sequence generated by fixed-point iteration. If $\lim_{n \rightarrow \infty} p_n = P$, then P is a fixed point of $g(x)$.

Proof. If $\lim_{n \rightarrow \infty} p_n = P$, then $\lim_{n \rightarrow \infty} p_{n+1} = P$. It follows from this result, the continuity of g , and the relation $p_{n+1} = g(p_n)$ that

$$(2) \quad g(P) = g\left(\lim_{n \rightarrow \infty} p_n\right) = \lim_{n \rightarrow \infty} g(p_n) = \lim_{n \rightarrow \infty} p_{n+1} = P.$$

Therefore, P is a fixed point of $g(x)$. •

Example 2.2. Consider the convergent iteration

$$p_0 = 0.5 \quad \text{and} \quad p_{k+1} = e^{-p_k} \quad \text{for } k = 0, 1, \dots$$

The first 10 terms are obtained by the calculations

$$\begin{aligned} p_1 &= e^{-0.500000} = 0.606531 \\ p_2 &= e^{-0.606531} = 0.545239 \\ p_3 &= e^{-0.545239} = 0.579703 \\ &\vdots \\ p_9 &= e^{-0.566409} = 0.567560 \\ p_{10} &= e^{-0.567560} = 0.566907 \end{aligned}$$

The sequence is converging, and further calculations reveal that

$$\lim_{n \rightarrow \infty} p_n = 0.567143 \dots$$

Thus we have found an approximation for the fixed point of the function $y = e^{-x}$. ■

The following two theorems establish conditions for the existence of a fixed point and the convergence of the fixed-point iteration process to a fixed point.

Theorem 2.2. Assume that $g \in C[a, b]$.

- (3) If the range of the mapping $y = g(x)$ satisfies $y \in [a, b]$ for all $x \in [a, b]$, then g has a fixed point in $[a, b]$.
- (4) Furthermore, suppose that $g'(x)$ is defined over (a, b) and that a positive constant $K < 1$ exists with $|g'(x)| \leq K < 1$ for all $x \in (a, b)$; then g has a unique fixed point P in $[a, b]$.

Proof of (3). If $g(a) = a$ or $g(b) = b$, the assertion is true. Otherwise, the values of $g(a)$ and $g(b)$ must satisfy $g(a) \in (a, b]$ and $g(b) \in [a, b)$. The function $f(x) \equiv x - g(x)$ has the property that

$$f(a) = a - g(a) < 0 \quad \text{and} \quad f(b) = b - g(b) > 0.$$

Now apply Theorem 1.2, the intermediate value theorem, to $f(x)$, with the constant $L = 0$, and conclude that there exists a number P with $P \in (a, b)$ so that $f(P) = 0$. Therefore, $P = g(P)$ and P is the desired fixed point of $g(x)$.

Proof of (4). Now we must show that this solution is unique. By way of contradiction, let us make the additional assumption that there exist two fixed points P_1 and P_2 . Now apply Theorem 1.6, the mean value theorem, and conclude that there exists a number $d \in (a, b)$ so that

$$(5) \quad g'(d) = \frac{g(P_2) - g(P_1)}{P_2 - P_1}.$$

Next, use the facts that $g(P_1) = P_1$ and $g(P_2) = P_2$ to simplify the right side of equation (5) and obtain

$$g'(d) = \frac{P_2 - P_1}{P_2 - P_1} = 1.$$

But this contradicts the hypothesis in (4) that $|g'(x)| < 1$ over (a, b) , so it is not possible for two fixed points to exist. Therefore, $g(x)$ has a unique fixed point P in $[a, b]$ under the conditions given in (4). •

Example 2.3. Apply Theorem 2.2 to show rigorously that $g(x) = \cos(x)$ has a unique fixed point in $[0, 1]$.

Clearly, $g \in C[0, 1]$. Also, $g(x) = \cos(x)$ is a decreasing function on $[0, 1]$; thus its range on $[0, 1]$ is $[\cos(1), 1] \subseteq [0, 1]$. Thus condition (3) of Theorem 2.2 is satisfied and g has a fixed point in $[0, 1]$. Finally, if $x \in (0, 1)$, then $|g'(x)| = |-\sin(x)| = \sin(x) \leq \sin(1) < 0.8415 < 1$. Thus $K = \sin(1) < 1$, condition (4) of Theorem 2.2 is satisfied, and g has a unique fixed point in $[0, 1]$. ■

We can now state a theorem that can be used to determine whether the fixed-point iteration process given in (1) will produce a convergent or a divergent sequence.

Theorem 2.3 (Fixed-Point Theorem). Assume that (i) $g, g' \in C[a, b]$, (ii) K is a positive constant, (iii) $p_0 \in (a, b)$, and (iv) $g(x) \in [a, b]$ for all $x \in [a, b]$.

- (6) If $|g'(x)| \leq K < 1$ for all $x \in [a, b]$, then the iteration $p_n = g(p_{n-1})$ will converge to the unique fixed point $P \in [a, b]$. In this case, P is said to be an attractive fixed point.
- (7) If $|g'(x)| > 1$ for all $x \in [a, b]$, then the iteration $p_n = g(p_{n-1})$ will not converge to P . In this case, P is said to be a repelling fixed point and the iteration exhibits local divergence.

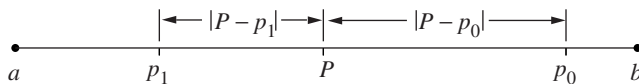


Figure 2.3 The relationship among P , p_0 , p_1 , $|P - p_0|$, and $|P - p_1|$.

Remark 1. It is assumed that $p_0 \neq P$ in statement (7).

Remark 2. Because g is continuous on an interval containing P , it is permissible to use the simpler criterion $|g'(P)| \leq K < 1$ and $|g'(P)| > 1$ in (6) and (7), respectively.

Proof. We first show that the points $\{p_n\}_{n=0}^{\infty}$ all lie in (a, b) . Starting with p_0 , we apply Theorem 1.6, the mean value theorem. There exists a value $c_0 \in (a, b)$ so that

$$(8) \quad \begin{aligned} |P - p_1| &= |g(P) - g(p_0)| = |g'(c_0)(P - p_0)| \\ &= |g'(c_0)||P - p_0| \leq K|P - p_0| < |P - p_0|. \end{aligned}$$

Therefore, p_1 is no further from P than p_0 was, and it follows that $p_1 \in (a, b)$ (see Figure 2.3). In general, suppose that $p_{n-1} \in (a, b)$; then

$$(9) \quad \begin{aligned} |P - p_n| &= |g(P) - g(p_{n-1})| = |g'(c_{n-1})(P - p_{n-1})| \\ &= |g'(c_{n-1})||P - p_{n-1}| \leq K|P - p_{n-1}| < |P - p_{n-1}|. \end{aligned}$$

Therefore, $p_n \in (a, b)$ and hence, by induction, all the points $\{p_n\}_{n=0}^{\infty}$ lie in (a, b) .

To complete the proof of (6), we will show that

$$(10) \quad \lim_{n \rightarrow \infty} |P - p_n| = 0.$$

First, a proof by induction will establish the inequality

$$(11) \quad |P - p_n| \leq K^n |P - p_0|.$$

The case $n = 1$ follows from the details in relation (8). Using the induction hypothesis $|P - p_{n-1}| \leq K^{n-1}|P - p_0|$ and the ideas in (9), we obtain

$$|P - p_n| \leq K|P - p_{n-1}| \leq K K^{n-1}|P - p_0| = K^n |P - p_0|.$$

Thus, by induction, inequality (11) holds for all n . Since $0 < K < 1$, the term K^n goes to zero as n goes to infinity. Hence

$$(12) \quad 0 \leq \lim_{n \rightarrow \infty} |P - p_n| \leq \lim_{n \rightarrow \infty} K^n |P - p_0| = 0.$$

The limit of $|P - p_n|$ is squeezed between zero on the left and zero on the right, so we can conclude that $\lim_{n \rightarrow \infty} |P - p_n| = 0$. Thus $\lim_{n \rightarrow \infty} p_n = P$ and, by Theorem 2.1, the iteration $p_n = g(p_{n-1})$ converges to the fixed point P . Therefore, statement (6) of Theorem 2.3 is proved. We leave statement (7) for the reader to investigate. •

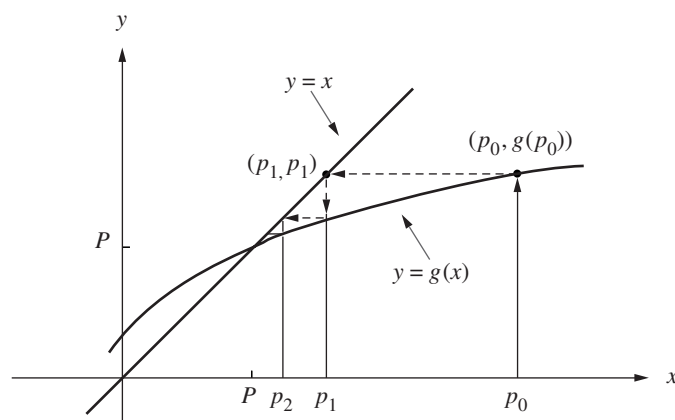


Figure 2.4 (a) Monotone convergence when $0 < g'(P) < 1$.

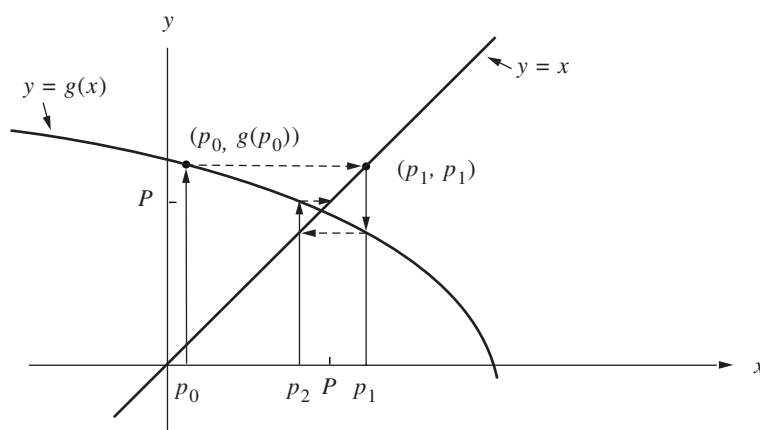


Figure 2.4 (b) Oscillating convergence when $-1 < g'(P) < 0$.

Corollary 2.1. Assume that g satisfies the hypothesis given in (6) of Theorem 2.3. Bounds for the error involved when using p_n to approximate P are given by

$$(13) \quad |P - p_n| \leq K^n |P - p_0| \quad \text{for all } n \geq 1$$

and

$$(14) \quad |P - p_n| \leq \frac{K^n |p_1 - p_0|}{1 - K} \quad \text{for all } n \geq 1.$$

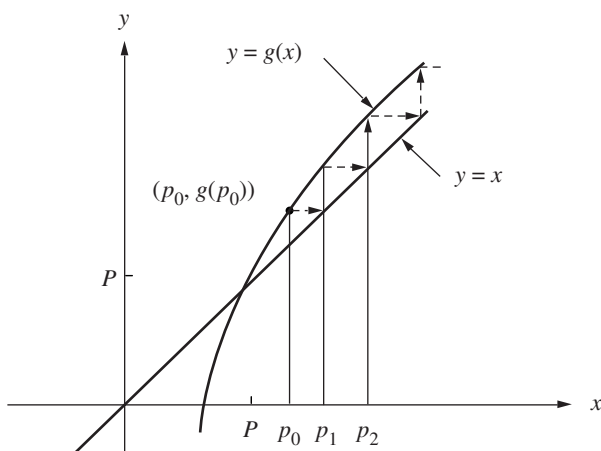


Figure 2.5 (a) Monotone divergence when $1 < g'(P)$.

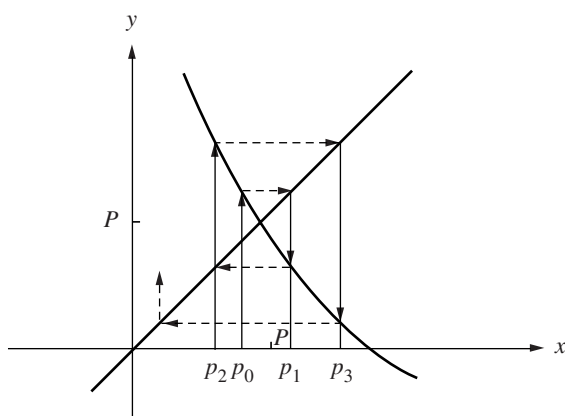


Figure 2.5 (b) Divergent oscillation when $g'(P) < -1$.

Graphical Interpretation of Fixed-Point Iteration

Since we seek a fixed point P to $g(x)$, it is necessary that the graph of the curve $y = g(x)$ and the line $y = x$ intersect at the point (P, P) . Two simple types of convergent iteration, monotone and oscillating, are illustrated in Figure 2.4(a) and (b), respectively.

To visualize the process, start at p_0 on the x -axis and move vertically to the point $(p_0, p_1) = (p_0, g(p_0))$ on the curve $y = g(x)$. Then move horizontally from (p_0, p_1) to the point (p_1, p_1) on the line $y = x$. Finally, move vertically downward to p_1 on the x -axis. The recursion $p_{n+1} = g(p_n)$ is used to construct the point (p_n, p_{n+1}) on the graph, then a horizontal motion locates (p_{n+1}, p_{n+1}) on the line $y = x$, and then a vertical movement ends up at p_{n+1} on the x -axis. The situation is shown in Figure 2.4.

If $|g'(P)| > 1$, then the iteration $p_{n+1} = g(p_n)$ produces a sequence that diverges away from P . The two simple types of divergent iteration, monotone and oscillating, are illustrated in Figure 2.5(a) and (b), respectively.

Example 2.4. Consider the iteration $p_{n+1} = g(p_n)$ when the function $g(x) = 1+x-x^2/4$ is used. The fixed points can be found by solving the equation $x = g(x)$. The two solutions (fixed points of g) are $x = -2$ and $x = 2$. The derivative of the function is $g'(x) = 1-x/2$, and there are only two cases to consider.

Case (i): $P = -2$
 Start with $p_0 = -2.05$
 then get $p_1 = -2.100625$
 $p_2 = -2.20378135$
 $p_3 = -2.41794441$

$$\lim_{n \rightarrow \infty} p_n = -\infty.$$

Since $|g'(x)| > \frac{3}{2}$ on $[-3, -1]$, by Theorem 2.3, the sequence will not converge to $P = -2$.

Case (ii): $P = 2$
 Start with $p_0 = 1.6$
 then get $p_1 = 1.96$
 $p_2 = 1.9996$
 $p_3 = 1.99999996$

$$\lim_{n \rightarrow \infty} p_n = 2.$$

Since $|g'(x)| < \frac{1}{2}$ on $[1, 3]$, by Theorem 2.3, the sequence will converge to $P = 2$. ■

Theorem 2.3 does not state what will happen when $g'(P) = 1$. The next example has been specially constructed so that the sequence $\{p_n\}$ converges whenever $p_0 > P$ and it diverges if we choose $p_0 < P$.

Example 2.5. Consider the iteration $p_{n+1} = g(p_n)$ when the function $g(x) = 2(x-1)^{1/2}$ for $x \geq 1$ is used. Only one fixed point $P = 2$ exists. The derivative is $g'(x) = 1/(x-1)^{1/2}$ and $g'(2) = 1$, so Theorem 2.3 does not apply. There are two cases to consider when the starting value lies to the left or right of $P = 2$.

Case (i): Start with $p_0 = 1.5$,
 then get $p_1 = 1.41421356$
 $p_2 = 1.28718851$
 $p_3 = 1.07179943$
 $p_4 = 0.53590832$
 \vdots
 $p_5 = 2(-0.46409168)^{1/2}$.

Since p_4 lies outside the domain of $g(x)$, the term p_5 cannot be computed.

Case (ii): Start with $p_0 = 2.5$,
 then get $p_1 = 2.44948974$
 $p_2 = 2.40789513$
 $p_3 = 2.37309514$
 $p_4 = 2.34358284$

$$\lim_{n \rightarrow \infty} p_n = 2.$$

This sequence is converging too slowly to the value $P = 2$; indeed, $P_{1000} = 2.00398714$. ■

Absolute and Relative Error Considerations

In Example 2.5, case (ii), the sequence converges slowly, and after 1000 iterations the three consecutive terms are

$$p_{1000} = 2.00398714, \quad p_{1001} = 2.00398317, \quad \text{and} \quad p_{1002} = 2.00397921.$$

This should not be disturbing; after all, we could compute a few thousand more terms and find a better approximation! But what about a criterion for stopping the iteration? Notice that if we use the difference between consecutive terms,

$$|p_{1001} - p_{1002}| = |2.00398317 - 2.00397921| = 0.00000396.$$

Yet the absolute error in the approximation p_{1000} is known to be

$$|P - p_{1000}| = |2.00000000 - 2.00398714| = 0.00398714.$$

This is about 1000 times larger than $|p_{1001} - p_{1002}|$ and it shows that closeness of consecutive terms does not guarantee that accuracy has been achieved. But it is usually the only criterion available and is often used to terminate an iterative procedure.

Numerical Methods Using Matlab, 4th Edition, 2004

John H. Mathews and Kurtis K. Fink

ISBN: 0-13-065248-2

Prentice-Hall Inc.

Upper Saddle River, New Jersey, USA

<http://vig.prenhall.com/>

NUMERICAL METHODS USING MATLAB

FOURTH EDITION



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