The coefficients a_j and b_j are computed with Euler's formulas:

(5)
$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(jx)\,dx \qquad \text{for } j = 0,\ 1,\ \ldots$$

and

(6)
$$b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(jx)\,dx \qquad \text{for } j = 1,\ 2,\ \ldots. \qquad \blacktriangle$$

The factor $\frac{1}{2}$ in the constant term $a_0/2$ in the Fourier series (4) has been introduced for convenience so that a_0 could be obtained from the general formula (5) by setting $j = 0$. Convergence of the Fourier series is discussed in the next result.

Theorem 5.5 (Fourier Expansion). Assume that $S(x)$ is the Fourier series for $f(x)$ over $[-\pi, \pi]$. If $f'(x)$ is piecewise continuous on $[-\pi, \pi]$ and has both a left- and right-hand derivative at each point in this interval, then $S(x)$ is convergent for all $x \in [-\pi, \pi]$. The relation

$$S(x) = f(x)$$

holds at all points $x \in [-\pi, \pi]$, where $f(x)$ is continuous. If $x = a$ is a point of discontinuity of f, then

$$S(a) = \frac{f(a^-) + f(a^+)}{2},$$

where $f(a^-)$ and $f(a^+)$ denote the left- and right-hand limits, respectively. With this understanding, we obtain the Fourier expansion:

(7)
$$f(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty}(a_j\cos(jx) + b_j\sin(jx)).$$

A brief outline of the derivation of formulas (5) and (6) is given at the end of the subsection.

Example 5.13. Show that the function $f(x) = x/2$ for $-\pi < x < \pi$, extended periodically by the equation $f(x + 2\pi) = f(x)$, has the Fourier series representation

$$f(x) = \sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}\sin(jx) = \sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \cdots.$$

Using Euler's formulas and integration by parts, we get

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{x}{2}\cos(jx)\,dx = \frac{x\sin(jx)}{2\pi j} + \frac{\cos(jx)}{2\pi j^2}\bigg|_{-\pi}^{\pi} = 0$$

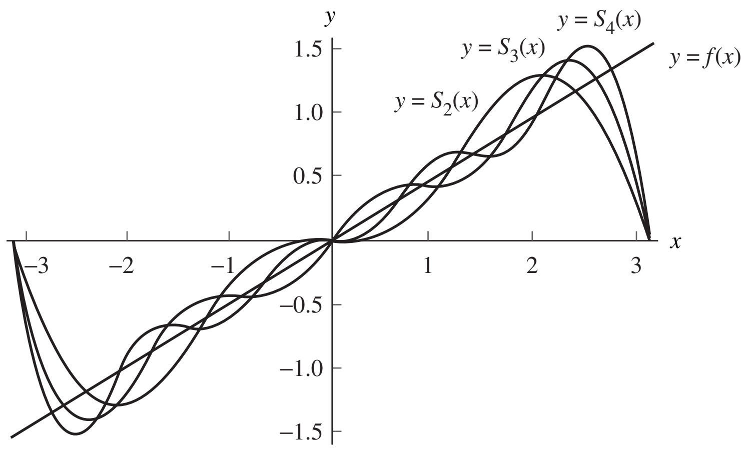

Figure 5.19 The function $f(x) = x/2$ over $[-\pi, \pi]$ and its trigonometric approximations $S_2(x)$, $S_3(x)$, and $S_4(x)$.

for $j = 1, 2, 3, \ldots$, and

$$b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x}{2}\sin(jx)\,dx = \frac{-x\cos(jx)}{2\pi j} + \frac{\sin(jx)}{2\pi j^2}\bigg|_{-\pi}^{\pi} = \frac{(-1)^{j+1}}{j}$$

for $j = 1, 2, 3, \ldots$. The coefficient a_0 is obtained by a separate calculation:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x}{2}\,dx = \frac{x^2}{4\pi}\bigg|_{-\pi}^{\pi} = 0.$$

These calculations show that all the coefficients of the cosine functions are zero. The graph of $f(x)$ and the partial sums

$$S_2(x) = \sin(x) - \frac{\sin(2x)}{2},$$
$$S_3(x) = \sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3},$$

and

$$S_4(x) = \sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \frac{\sin(4x)}{4}$$

are shown in Figure 5.19. ∎

We now state some general properties of Fourier series. The proofs are left as exercises.

Theorem 5.6 (Cosine Series). Suppose that $f(x)$ is an even function; that is, suppose that $f(-x) = f(x)$ holds for all x. If $f(x)$ has period 2π and if $f(x)$ and $f'(x)$ are piecewise continuous, then the Fourier series for $f(x)$ involves only cosine terms:

$$f(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos(jx), \tag{8}$$

where

$$a_j = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(jx)\, dx \qquad \text{for } j = 0,\ 1,\ \ldots. \tag{9}$$

Theorem 5.7 (Sine Series). Suppose that $f(x)$ is an odd function; that is, $f(-x) = -f(x)$ holds for all x. If $f(x)$ has period 2π and if $f(x)$ and $f'(x)$ are piecewise continuous, then the Fourier series for $f(x)$ involves only the sine terms:

$$f(x) = \sum_{j=1}^{\infty} b_j \sin(jx), \tag{10}$$

where

$$b_j = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(jx)\, dx \qquad \text{for } j = 1,\ 2,\ \ldots. \tag{11}$$

Example 5.14. Show that the function $f(x) = |x|$ for $-\pi < x < \pi$, extended periodically by the equation $f(x + 2\pi) = f(x)$, has the Fourier cosine representation

$$\begin{aligned} f(x) &= \frac{\pi}{2} - \frac{4}{\pi} \sum_{j=1}^{\infty} \frac{\cos((2j-1)x)}{(2j-1)^2} \\ &= \frac{\pi}{2} - \frac{4}{\pi} \left(\cos(x) + \frac{\cos(3x)}{3^2} + \frac{\cos(5x)}{5^2} + \cdots \right). \end{aligned} \tag{12}$$

The function $f(x)$ is an even function, so we can use Theorem 5.6 and need only to compute the coefficients $\{a_j\}$:

$$\begin{aligned} a_j &= \frac{2}{\pi} \int_0^{\pi} x \cos(jx)\, dx = \frac{2x \sin(jx)}{\pi j} + \frac{2\cos(jx)}{\pi j^2} \Big|_0^{\pi} \\ &= \frac{2\cos(j\pi) - 2}{\pi j^2} = \frac{2((-1)^j - 1)}{\pi j^2} \qquad \text{for } j = 1,\ 2,\ 3,\ \ldots. \end{aligned}$$

Since $((-1)^j - 1) = 0$ when j is even, the cosine series will involve only the odd terms. The odd coefficients have the pattern

$$a_1 = \frac{-4}{\pi}, \qquad a_3 = \frac{-4}{\pi 3^2}, \qquad a_5 = \frac{-4}{\pi 5^2}, \qquad \ldots.$$

The coefficient a_0 is obtained by the separate calculation

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{x^2}{\pi}\bigg|_0^{\pi} = \pi.$$

Therefore, we have found the desired coefficients in (12). ■

Proof of Euler's Formulas for Theorem 5.5. The following heuristic argument assumes the existence and convergence of the Fourier series representation. To determine a_0, we can integrate both sides of (7) and get

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\,dx &= \int_{-\pi}^{\pi}\left(\frac{a_0}{2} + \sum_{j=1}^{\infty}(a_j\cos(jx) + b_j\sin(jx))\right)dx \\ &= \int_{-\pi}^{\pi}\frac{a_0}{2}\,dx + \sum_{j=1}^{\infty}a_j\int_{-\pi}^{\pi}\cos(jx)\,dx + \sum_{j=1}^{\infty}b_j\int_{-\pi}^{\pi}\sin(jx)\,dx \\ &= \pi a_0 + 0 + 0.\end{aligned} \tag{13}$$

Justification for switching the order of integration and summation requires a detailed treatment of uniform convergence and can be found in advanced texts. Hence we have shown that

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx. \tag{14}$$

To determine a_m, we let $m > 0$ be a fixed integer, multiply both sides of (7) by $\cos(mx)$, and integrate both sides to obtain

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos(mx)\,dx + \sum_{j=1}^{\infty}a_j\int_{-\pi}^{\pi}\cos(jx)\cos(mx)\,dx \\ + \sum_{j=1}^{\infty}b_j\int_{-\pi}^{\pi}\sin(jx)\cos(mx)\,dx.\end{aligned} \tag{15}$$

Equation (15) can be simplified by using the orthogonal properties of the trigonometric functions, which are now stated. The value of the first term on the right-hand side of (15) is

$$\frac{a_0}{2}\int_{-\pi}^{\pi}\cos(mx)\,dx = \frac{a_0\sin(mx)}{2m}\bigg|_{-\pi}^{\pi} = 0. \tag{16}$$

The value of the term involving $\cos(jx)\cos(mx)$ is found by using the trigonometric identity

$$\cos(jx)\cos(mx) = \frac{1}{2}\cos((j+m)x) + \frac{1}{2}\cos((j-m)x). \tag{17}$$

When $j \neq m$, then (17) is used to get

$$a_j \int_{-\pi}^{\pi} \cos(jx)\cos(mx)\,dx = \frac{1}{2}a_j \int_{-\pi}^{\pi} \cos((j+m)x)\,dx + \frac{1}{2}a_j \int_{-\pi}^{\pi} \cos((j-m)x)\,dx = 0 + 0 = 0. \tag{18}$$

When $j = m$, the value of the integral is

$$a_m \int_{-\pi}^{\pi} \cos(jx)\cos(mx)\,dx = a_m \pi. \tag{19}$$

The value of the term on the right side of (15) involving $\sin(jx)\cos(mx)$ is found by using the trigonometric identity

$$\sin(jx)\cos(mx) = \frac{1}{2}\sin((j+m)x) + \frac{1}{2}\sin((j-m)x). \tag{20}$$

For all values of j and m in (20), we obtain

$$b_j \int_{-\pi}^{\pi} \sin(jx)\cos(mx)\,dx = \frac{1}{2}b_j \int_{-\pi}^{\pi} \sin((j+m)x)\,dx + \frac{1}{2}b_j \int_{-\pi}^{\pi} \sin((j-m)x)\,dx = 0 + 0 = 0. \tag{21}$$

Therefore, using the results of (16), (18), (19), and (21) in equation (15), we conclude that

$$\pi a_m = \int_{-\pi}^{\pi} f(x)\cos(mx)\,dx, \qquad \text{for } m = 1, 2, \ldots. \tag{22}$$

Therefore, Euler's formula (5) is established. Euler's formula (6) is proved similarly. •

Trigonometric Polynomial Approximation

Definition 5.4. A series of the form

$$T_M(x) = \frac{a_0}{2} + \sum_{j=1}^{M}(a_j\cos(jx) + b_j\sin(jx)) \tag{23}$$

is called a ***trigonometric polynomial*** of order M. ▲

Theorem 5.8 (Discrete Fourier Series). Suppose that $\{(x_j, y_j)\}_{j=0}^{N}$ are $N+1$ points, where $y_j = f(x_j)$, and the abscissas are equally spaced:

$$x_j = -\pi + \frac{2j\pi}{N} \quad \text{for } j = 0, 1, \ldots, N. \tag{24}$$

If $f(x)$ is periodic with period 2π and $2M < N$, then there exists a trigonometric polynomial $T_M(x)$ of the form (23) that minimizes the quantity

$$\sum_{k=1}^{N} (f(x_k) - T_M(x_k))^2. \tag{25}$$

The coefficients a_j and b_j of this polynomial are computed with the formulas

$$a_j = \frac{2}{N} \sum_{k=1}^{N} f(x_k) \cos(jx_k) \quad \text{for } j = 0, 1, \ldots, M, \tag{26}$$

and

$$b_j = \frac{2}{N} \sum_{k=1}^{N} f(x_k) \sin(jx_k) \quad \text{for } j = 1, 2, \ldots, M. \tag{27}$$

Although formulas (26) and (27) are defined with the least-squares procedure, they can also be viewed as numerical approximations to the integrals in Euler's formulas (5) and (6). Euler's formulas give the coefficients for the Fourier series of a continuous function, whereas formulas (26) and (27) give the trigonometric polynomial coefficients for curve fitting to data points. The next example uses data points generated by the function $f(x) = x/2$ at discrete points. When more points are used, the trigonometric polynomial coefficients get closer to the Fourier series coefficients.

Example 5.15. Use the 12 equally spaced points $x_k = -\pi + k\pi/6$, for $k = 1, 2, \ldots, 12$, and find the trigonometric polynomial approximation for $M = 5$ to the 12 data points $\{(x_k, f(x_k))\}_{k=1}^{12}$, where $f(x) = x/2$. Also compare the results when 60 and 360 points are used and with the first five terms of the Fourier series expansion for $f(x)$ that is given in Example 5.13.

Since the periodic extension is assumed, at a point of discontinuity, the function value $f(\pi)$ must be computed using the formula

$$f(\pi) = \frac{f(\pi^-) + f(\pi^+)}{2} = \frac{\pi/2 - \pi/2}{2} = 0. \tag{28}$$

The function $f(x)$ is an odd function; hence the coefficients for the cosine terms are all zero (i.e., $a_j = 0$ for all j). The trigonometric polynomial of degree $M = 5$ involves only the sine terms, and when formula (27) is used with (28), we get

$$\begin{aligned} T_5(x) = {} & 0.9770486 \sin(x) - 0.4534498 \sin(2x) + 0.26179938 \sin(3x) \\ & - 0.1511499 \sin(4x) + 0.0701489 \sin(5x). \end{aligned} \tag{29}$$

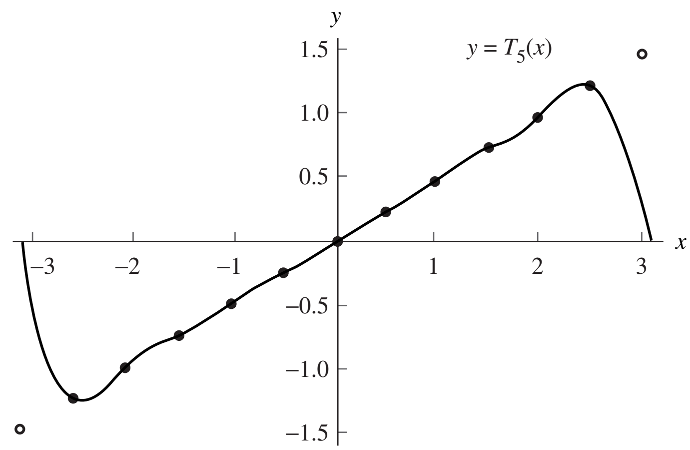

Figure 5.20 The trigonometric polynomial $T_5(x)$ of degree $M = 5$, based on 12 data points that lie on the line $y = x/2$.

Table 5.9 Comparison of Trigonometric Polynomial Coefficients for Approximations to $f(x) = x/2$ over $[-\pi, \pi]$

	Trigonometric polynomial coefficients			Fourier series coefficients
	12 points	60 points	360 points	
b_1	0.97704862	0.99908598	0.99997462	1.0
b_2	−0.45344984	−0.49817096	−0.49994923	−0.5
b_3	0.26179939	0.33058726	0.33325718	0.33333333
b_4	−0.15114995	−0.24633386	−0.24989845	−0.25
b_5	0.07014893	0.19540972	0.19987306	0.2

The graph of $T_5(x)$ is shown in Figure 5.20.

The coefficients of the fifth-degree trigonometric polynomial change slightly when the number of interpolation points increases to 60 and 360. As the number of points increases, they get closer to the coefficients of the Fourier series expansion of $f(x)$. The results are compared in Table 5.9. ■

Numerical Methods Using Matlab, 4th Edition, 2004
John H. Mathews and Kurtis K. Fink
ISBN: 0-13-065248-2

Prentice-Hall Inc.
Upper Saddle River, New Jersey, USA
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