Linear Shooting Method

9.8 Boundary Value Problems

Another type of differential equation has the form

$$x'' = f(t, x, x') \quad \text{for } a \le t \le b, \tag{1}$$

with the boundary conditions

$$x(a) = \alpha \quad \text{and} \quad x(b) = \beta. \tag{2}$$

This is called a ***boundary value problem***.

The conditions that guarantee that a solution to (1) exists should be checked before any numerical scheme is applied; otherwise, a list of meaningless output may be generated. The general conditions are stated in the following theorem.

Theorem 9.8 (Boundary Value Problem). Assume that $f(t, x, y)$ is continuous on the region $R = \{(t, x, y) : a \leq t \leq b, -\infty < x < \infty, -\infty < y < \infty\}$ and that $\partial f/\partial x = f_x(t, x, y)$ and $\partial f/\partial y = f_y(t, x, y)$ are continuous on R. If there exists a constant $M > 0$ for which f_x and f_y satisfy

$$f_x(t, x, y) > 0 \quad \text{for all } (t, x, y) \in R \text{ and} \tag{3}$$

$$|f_y(t, x, y)| \leq M \quad \text{for all } (t, x, y) \in R, \tag{4}$$

then the boundary value problem

$$x'' = f(t, x, x') \quad \text{with } x(a) = \alpha \text{ and } x(b) = \beta \tag{5}$$

has a unique solution $x = x(t)$ for $a \leq t \leq b$.

The notation $y = x'(t)$ has been used to distinguish the third variable of the function $f(t, x, x')$. Finally, the special case of linear differential equations is worthy of mention.

Corollary 9.1 (Linear Boundary Value Problem). Assume that f in Theorem 9.8 has the form $f(t, x, y) = p(t)y + q(t)x + r(t)$ and that f and its partial derivatives $\partial f/\partial x = q(t)$ and $\partial f/\partial y = p(t)$ are continuous on R. If there exists a constant $M > 0$ for which $p(t)$ and $q(t)$ satisfy

$$q(t) > 0 \quad \text{for all } t \in [a, b] \tag{6}$$

and

$$|p(t)| \leq M = \max_{a \leq t \leq b} \{|p(t)|\}, \tag{7}$$

then the ***linear boundary value problem***

$$x'' = p(t)x'(t) + q(t)x(t) + r(t) \quad \text{with } x(a) = \alpha \text{ and } x(b) = \beta \tag{8}$$

has a unique solution $x = x(t)$ over $a \leq t \leq b$.

Reduction to Two I.V.P.s: Linear Shooting Method

Finding the solution of a linear boundary problem is assisted by the linear structure of the equation and the use of two special initial value problems. Suppose that $u(t)$ is the unique solution to the I.V.P.

$$u'' = p(t)u'(t) + q(t)u(t) + r(t) \quad \text{with } u(a) = \alpha \text{ and } u'(a) = 0. \tag{9}$$

Furthermore, suppose that $v(t)$ is the unique solution to the I.V.P.

$$v'' = p(t)v'(t) + q(t)v(t) \quad \text{with } v(a) = 0 \text{ and } v'(a) = 1. \tag{10}$$

Then the linear combination

$$x(t) = u(t) + Cv(t) \tag{11}$$

is a solution to $x'' = p(t)x'(t) + q(t)x(t) + r(t)$ as seen by the computation

$$\begin{aligned} x'' = u'' + Cv'' &= p(t)u'(t) + q(t)u(t) + r(t) + p(t)Cv'(t) + q(t)Cv(t) \\ &= p(t)(u'(t) + Cv'(t)) + q(t)(u(t) + Cv(t)) + r(t) \\ &= p(t)x'(t) + q(t)x(t) + r(t). \end{aligned}$$

The solution $x(t)$ in equation (11) takes on the boundary values

$$\begin{aligned} x(a) &= u(a) + Cv(a) = \alpha + 0 = \alpha, \\ x(b) &= u(b) + Cv(b). \end{aligned} \tag{12}$$

Imposing the boundary condition $x(b) = \beta$ in (12) produces $C = (\beta - u(b))/v(b)$. Therefore, if $v(b) \neq 0$, the unique solution to (8) is

$$x(t) = u(t) + \frac{\beta - u(b)}{v(b)} v(t). \tag{13}$$

Remark. If q fulfills the hypotheses of Corollary 9.1, this rules out the troublesome solution $v(t) \equiv 0$, so that (13) is the form of the required solution. The details are left for the reader to investigate in the exercises.

Example 9.17. Solve the boundary value problem

$$x''(t) = \frac{2t}{1+t^2} x'(t) - \frac{2}{1+t^2} x(t) + 1$$

with $x(0) = 1.25$ and $x(4) = -0.95$ over the interval [0, 4].

The functions p, q, and r are $p(t) = 2t/(1 + t^2)$, $q(t) = -2/(1 + t^2)$, and $r(t) = 1$, respectively. The Runge-Kutta method of order 4 with step size $h = 0.2$ is used to construct numerical solutions $\{u_j\}$ and $\{v_j\}$ to equations (9) and (10), respectively. The approximations $\{u_j\}$ for $u(t)$ are given in the first column of Table 9.15. Then $u(4) \approx u_{20} = -2.893535$ and $v(4) \approx v_{20} = 4$ are used with (13) to construct

$$w_j = \frac{b - u(4)}{v(4)} v_j = 0.485884 v_j.$$

Table 9.15 Approximate Solutions $\{x_j\} = \{u_j + w_j\}$ to the Equation $x''(t) = \dfrac{2t}{1+t^2}x'(t) - \dfrac{2}{1+t^2} + 1$

t_j	u_j	w_j	$x_j = u_j + w_j$
0.0	1.250000	0.000000	1.250000
0.2	1.220131	0.097177	1.317308
0.4	1.132073	0.194353	1.326426
0.6	0.990122	0.291530	1.281652
0.8	0.800569	0.388707	1.189276
1.0	0.570844	0.485884	1.056728
1.2	0.308850	0.583061	0.891911
1.4	0.022522	0.680237	0.702759
1.6	−0.280424	0.777413	0.496989
1.8	−0.592609	0.874591	0.281982
2.0	−0.907039	0.971767	0.064728
2.2	−1.217121	1.068944	−0.148177
2.4	−1.516639	1.166121	−0.350518
2.6	−1.799740	1.263297	−0.536443
2.8	−2.060904	1.360474	−0.700430
3.0	−2.294916	1.457651	−0.837265
3.2	−2.496842	1.554828	−0.942014
3.4	−2.662004	1.652004	−1.010000
3.6	−2.785960	1.749181	−1.036779
3.8	−2.864481	1.846358	−1.018123
4.0	−2.893535	1.943535	−0.950000

Then the required approximate solution is $\{x_j\} = \{u_j + w_j\}$. Sample computations are given in Table 9.15, and Figure 9.24 shows their graphs. The reader can verify that $v(t) = t$ is the analytic solution for boundary value problem (10); that is,

$$v''(t) = \frac{2t}{1+t^2}v'(t) - \frac{2}{1+t^2}v(t)$$

with the initial conditions $v(0) = 0$ and $v'(0) = 1$.

The approximations in Table 9.16 compare numerical solutions obtained with the linear shooting method with the step sizes $h = 0.2$ and $h = 0.1$ and the analytic solution

$$x(t) = 1.25 + 0.4860896526t - 2.25t^2 + 2t\arctan(t) - \frac{1}{2}\ln(1+t^2) + \frac{1}{2}t^2\ln(1+t^2).$$

A graph of the approximate solution when $h = 0.2$ is given in Figure 9.25. Included in the table are columns for the error. Since the Runge-Kutta solutions have error of order $\boldsymbol{O}(h^4)$, the error in the solution with the smaller step size $h = 0.1$ is about $\frac{1}{16}$ the error of the solution with the large step size $h = 0.2$. ■

Program 9.10 will call Program 9.9 to solve the initial value problems (9) and (10). Program 9.9 approximates solutions of systems of differential equations using a modification of the Runge-Kutta method of order $N = 4$. Thus, it is necessary to save

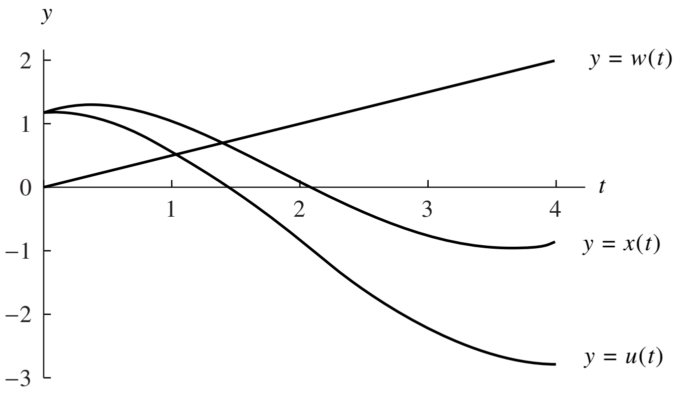

Figure 9.24 The numerical approximations $u(t)$ and $w(t)$ used to form $x(t) = u(t) + w(t)$, which is the solution to

$$x''(t) = \frac{2t}{1+t^2}x'(t) - \frac{2}{1+t^2}x(t) + 1.$$

Table 9.16 Numerical Approximations for $x''(t) = \dfrac{2t}{1+t^2}x'(t) - \dfrac{2}{1+t^2}x(t) + 1$

t_j	x_j $h = 0.2$	$x(t_j)$ exact	$x(t_j) - x_j$ error	t_j	x_j $h = 0.1$	$x(t_j)$ exact	$x(t_j) - x_j$ error
0.0	1.250000	1.250000	0.000000	0.0	1.250000	1.250000	0.000000
				0.1	1.291116	1.291117	0.000001
0.2	1.317308	1.317350	0.000042	0.2	1.317348	1.317350	0.000002
				0.3	1.328986	1.328990	0.000004
0.4	1.326426	1.326505	0.000079	0.4	1.326500	1.326505	0.000005
				0.5	1.310508	1.310514	0.000006
0.6	1.281652	1.281762	0.000110	0.6	1.281756	1.281762	0.000006
0.8	1.189276	1.189412	0.000136	0.8	1.189404	1.189412	0.000008
1.0	1.056728	1.056886	0.000158	1.0	1.056876	1.056886	0.000010
1.2	0.891911	0.892086	0.000175	1.2	0.892076	0.892086	0.000010
1.6	0.496989	0.497187	0.000198	1.6	0.497175	0.497187	0.000012
2.0	0.064728	0.064931	0.000203	2.0	0.064919	0.064931	0.000012
2.4	−0.350518	−0.350325	0.000193	2.4	−0.350337	−0.350325	0.000012
2.8	−0.700430	−0.700262	0.000168	2.8	−0.700273	−0.700262	0.000011
3.2	−0.942014	−0.941888	0.000126	3.2	−0.941895	−0.941888	0.000007
3.6	−1.036779	−1.036708	0.000071	3.6	−1.036713	−1.036708	0.000005
4.0	−0.950000	−0.950000	0.000000	4.0	−0.950000	−0.950000	0.000000

the equations (9) and (10) in the form of the system of equations (11) of Section 9.7.

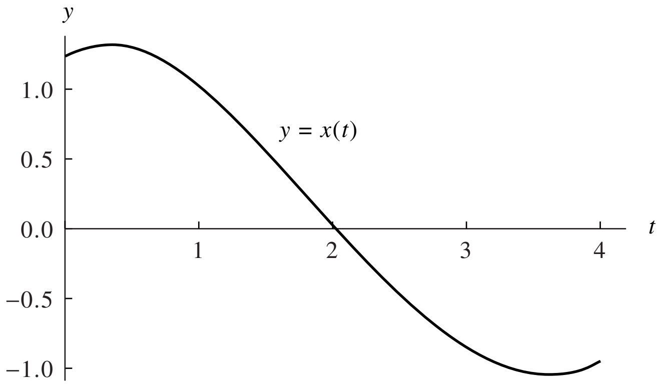

Figure 9.25 The graph of the numerical approximation for

$$x''(t) = \frac{2t}{1+t^2}x'(t) - \frac{2}{1+t^2}x(t) + 1$$

(using $h = 0.2$).

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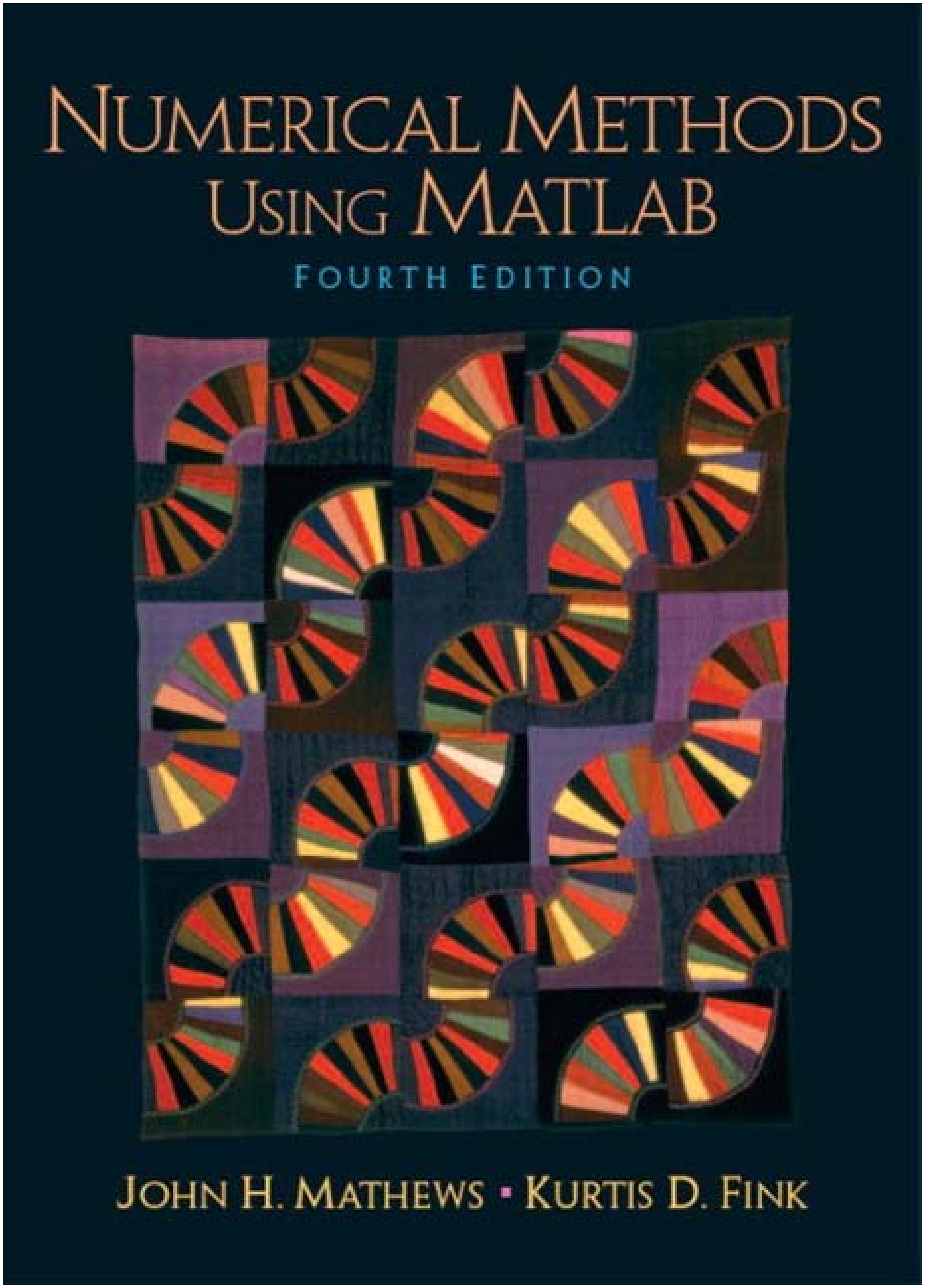

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